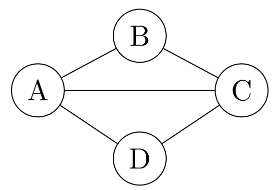
Integer Linear Programming

CS 181 Advanced Algorithms — Fall 2025

Maximum Independent Set

Warm-up: Below is a small example graph. Circle as many vertices as you can so that no two are connected. This is called an independent set.



Definition. Given a graph $G = (V, E)$, an *independent set* is a subset of vertices $S \subseteq V$ such that no two vertices in S share an edge. The goal is to find the largest such set.

Tasks:

- Formulate the problem as an ILP. Do the integer solutions of your ILP correspond to valid independent sets?
- What happens if you relax $x_v \in [0, 1]$? Can you find a fractional solution that beats all integral ones on a small graph?
- What is the integrality gap of your formulation?
- Is there a way to strengthen your formulation by adding more constraints?

3-SAT

Warm up: Consider the boolean formula $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$. The $\vee$ represents OR and the $\wedge$ represents AND. Can you find an assignment of true/false to each variable so that the formula becomes True?

Definition. In 3-SAT, we're given a Boolean formula in Conjunctive Normal Form where each clause contains exactly three literals (variables or their negations). The goal is to determine if there is a truth assignment satisfying all clauses.

Tasks:

- Represent this decision problem as an ILP using binary variables for the literals. Note that this is a decision problem, so your model may not need an objective function.
- Is there always a feasible fractional solution?
- If instead you wanted to *maximize* the number of satisfied clauses, how could you modify your model?
- What is the integrality gap of your optimization model?

Min-Cost Bipartite Perfect Matching (MCBPM)

Definition. Given a bipartite graph $G = (L, R, E)$ where $|L| = |R|$ and each edge (i, j) has a cost c_{ij}, a *perfect matching* picks exactly one edge incident to each vertex. The goal is to minimize total cost.

Tasks:

- Formulate this problem as an integer program.
- A note: as we have seen, this particular class of integer programs is solvable in polynomial time via the Hungarian algorithm
- Consider the linear relaxation (with $x_{ij} \in [0, 1]$). Can you find an instance of MCBPM with a fractional solution which costs less than the cheapest integral matching?

Vertex Cover

Warm-up: Using the same graph from Independent Set, find the smallest set of vertices that touches every edge.

Definition. Given $G = (V, E)$, a *vertex cover* is a set $S \subseteq V$ such that every edge in E has at least one endpoint in S. The goal is to find a vertex cover of minimum size.

Tasks:

- Formulate this problem as an ILP.
- Relax your model ($x_v \in [0, 1]$) and compute the fractional optimum on a small example. What is the integrality gap?
- Compare your relaxation with the Max-Independent Set relaxation — how are they related?